On point separation by arrangements of lines

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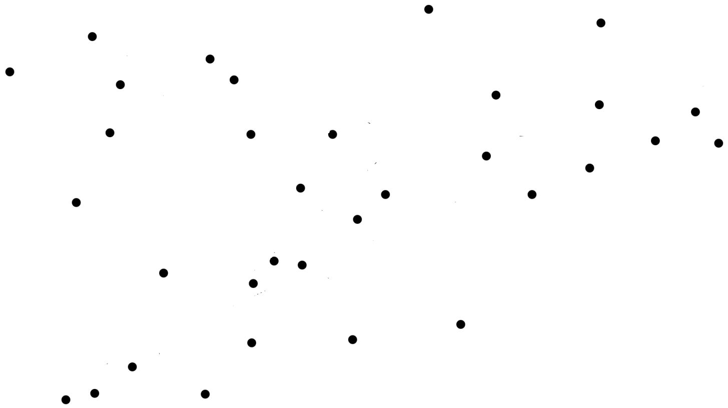
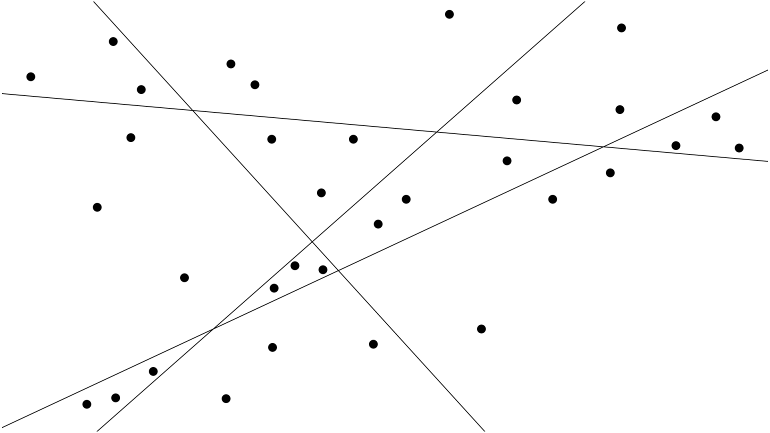
Problem

- We have a set X of N points and some integer D.
- Can we draw D lines such that each cell contains the same number of points?

Problem



Problem



Problem

- An arrangement of D lines divides the plane into $\approx D^2$ cells.
- An equal separation means each cell has $\approx D^{-2}N$ points.
- If we find a cell with more points, then the set is not separated equally.

Problem

For an arrangement of D lines A_D, consider the maximum number of points of X that lie in a single cell. Then the cutting number is the minimum of this value over all A_D:

$$\mathrm{CUT_D(X)} = \min_{A_D}(\max_{\text{cells}}|X \bigcap \text{cell}|).$$

Problem

Can X be divided evenly among the cells defined by some arrangement?

Problem

This is where our inquiry begins. Our problem and results are related to various important concepts in mathematics.

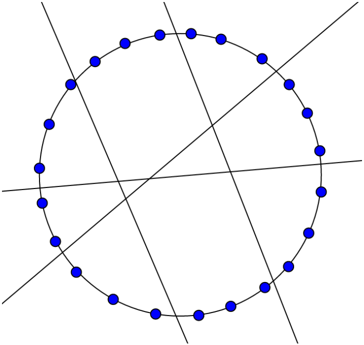
- The Szemeredi-Trotter Theorem in incidence geometry.
- Stabbing numbers in computational geometry.
- The Erdos-Szekeres Conjecture in Ramsey theory.

Problem

- We have $2D$ variables and $\approx D^2$ conditions to satisfy.
- Seems unlikely to always be possible.

Counterexample

Suppose X lies on convex curve Y and an arrangement of D lines is added.



Counterexample

- Each line intersects Y at most 2 times.
- Y is broken into at most $2D$ parts.
- Each is contained in a different cell.
- $\mathrm{CUT_D(X)} \geq \lceil \frac{N}{2D} \rceil > \frac{N}{D^2}$
- X cannot be evenly separated.

Convexity

- $\mathrm{CUT_D(X)}$ is large when X lies on a convex curve.
- The existence of a large convex subset is an obstacle to equal separation.
- We can show a convex curve gives the largest $\mathrm{CUT_D(X)}$.

Upper bound on cutting number

Theorem

If $D \geq \lceil \frac{N}{2k} \rceil$, *then* $\mathrm{CUT_D(X)} \leq k$.

Upper bound on cutting number

Corollary

$\mathrm{CUT_D(X)} \le \lceil \frac{N}{2D} \rceil$ *for any set* X.

Corollary

If X *lies on some convex curve,* $\mathrm{CUT_D(X)} = \lceil \frac{N}{2D} \rceil$.

Convexity

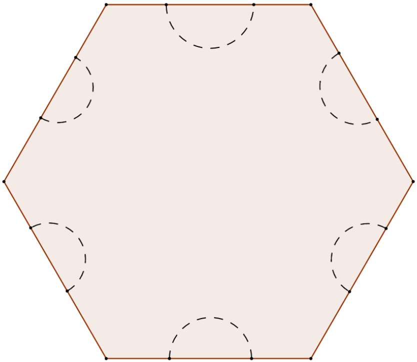
The convex number of X is the maximum number of points that lie on some convex curve in the plane:

$$\text{CON(X)} = \max_{\text{convex curves}} |X \bigcap \text{curve}|.$$

Convexity

- $\mathrm{CUT_D}(\mathrm{X}) \geq \frac{\mathrm{CON}(\mathrm{X})}{2D}$.
- If CON(X) is large, then $\mathrm{CUT_D}(\mathrm{X})$ is large.
- Does a large $\mathrm{CUT_D}(\mathrm{X})$ imply a large CON(X)?

Convexity



Convexity

Proposition

There exists a set of N points X such that $\mathrm{CUT_D(X)} \geq \frac{N}{4D}$ *and* $\mathrm{CON(X)} \leq 4\sqrt[3]{N}$.

Convexity

- Convexity is not the only obstacle to equal separation.
- We must find some other classification method.

Curves defined by stabbing number

Definition

The **stabbing number** of a geometric object is defined as the maximum number of intersections between it and any line in the plane.

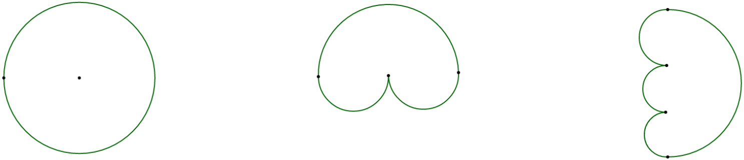


Figure: Stabbing number of 2, 4, and 6 respectively.

Curves defined by stabbing number

Definition

A **d-curve** is a curve with a stabbing number of d or less.

Definition

The **degree d(X)** of X is the minimum d for which a d-curve contains X.

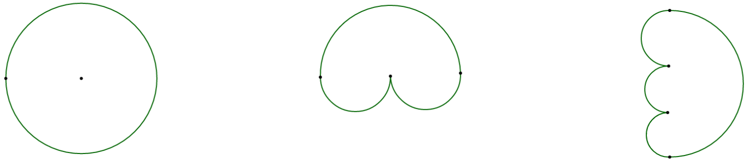


Figure: 2-curve, 4-curve, and 6-curve respectively.

Curves defined by stabbing number

- We have that $\mathrm{CUT}_{\mathrm{D}}(\mathrm{X}) \geq \frac{N}{d(X) \cdot D}$.

Curves defined by stabbing number

- A small $d(X)$ means a large $\mathrm{CUT_D(X)}$.
- If X can be separated equally, $d(X)$ is large.
- Considering all subsets of X, we have

$$\mathrm{CUT_D(X)} \geq \max_{Y \subseteq X} \frac{|Y|}{d(Y) \cdot D} \geq \frac{N}{d(X) \cdot D}$$

- We seek to establish the properties of $d(X)$ and d-curves.

Properties of degree

For fixed N, how big can $d(X)$ be?

Properties of degree

Definition

Let $D(N)$ be the maximum degree of any set of N points.

Examples

- $D(1) = D(2) = D(3) = 2$
- $D(4) = D(5) = D(6) = 4$

Properties of degree

Theorem

We can show that $D(N) = \Theta(\sqrt{N})$.

Properties of degree

- Existing literature shows there exists a simple spanning tree with stabbing number $O(\sqrt{N})$.
- We can convert this tree into a d-curve without increasing the order of the stabbing number.
- We can show that the bound is tight.

Properties of d-curves

Lemma

Let A and B be non-intersecting curves consisting of a finite number of points. Then there exists a curve Y through all the points of A and B such that $d(Y) \leq d(A) + d(B) + 2$.

Proposition

Suppose we have k non-intersecting curves $Y_1, Y_2, ..., Y_k$ consisting of a finite number of points. Then curve Z containing all points of Y_i has $d(Z) \leq 2D(k) + \sum_i d(Y_i)$.

Erdos-Szekeres problem on points in convex position

- What is $ES(n)$, the smallest number of points that must contain a convex curve with n points?
- The Erdos-Szekeres conjecture postulates $ES(n) = 2^{n-2} + 1$.
- However for 80 years, $ES(n) \leq 4^{n-o(n)}$.
- In April 2016, Andrew Suk gave $ES(n) \leq 2^{n+o(n)}$.

Existence of n points on a d-curve

We can generalize this problem to finding n points on a d curve.

Definition

Let $F(n, d)$ be the smallest number of points needed for a set to contain a d-curve with n points.

All convex curves are d-curves, so $F(n, d) \leq F(n, 2) = ES(n) \leq 2^{n+o(n)}$.

Existence of n points on a d-curve

Theorem

For arbitrarily large fixed d, we have $F(n, d) \leq 4^{n/d+o(n)}$.

- Note how $d = 2$ gives the bound for $ES(n)$.
- The proof involves the splitting method used for finding the upper bound on $\mathrm{CUT_D(X)}$.
- It also utilizes the previous lemma involving unions of d-curves.

Future research

- Does there exist some k such that if all subsets of k points in X lie on a d-curve then X lies on a d-curve?
- How can we use the geometry of X to find tighter upper bounds on $\mathrm{CUT}_{\mathrm{D}}(\mathrm{X})$?
- Is degree the only obstacle to separating a set of points equally?

Acknowledgments

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